Statistical Modelling for Dating Ice Cores

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- Introduction to Ice Core Dating
  - Ice Cores as Archives
  - Existing Dating methods

- A Bayesian Approach to Glaciological Modelling
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Introduction to Ice Core Dating

- Ice Cores – The Archive
- Existing Dating Methods

A Bayesian Approach to Glaciological Modelling

The Dating Uncertainty

Discussion
Ice Cores – The Archive

- Preserve valuable information about the climate and environment of the past
- Record chemical composition of snow, dust and atmospheric gases with high resolution for up to 700,000 years and longer [Parrenin et al., 2007]

Source: BAS image database
Ice Cores – The Archive

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- Record chemical composition of snow, dust and atmospheric gases with high resolution for up to 700,000 years and longer [Parrenin et al., 2007]
- Dating is essential to interpret this information
- Dating: relate time to depth

![Isotopic Content of Ice vs Snow Depth](Source: BAS image database)
Ice Cores – The Archive

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- Record chemical composition of snow, dust and atmospheric gases with high resolution for up to 700,000 years and longer [Parrenin et al., 2007]
- Dating is essential to interpret this information
- Dating: relate time to depth

Source: BAS image database
Existing Dating Methods

- layer counting using seasonality in signals

- glaciological modelling
  - model of accumulation: estimated from isotopic content of ice
  - model of mechanical processes after accumulation: i.e. snow densification, ice flow

- comparison with other dated records
  - e.g. ice cores, volcanic eruptions, insolation changes

- any combination of dating methods
Existing Dating Methods

- layer counting using seasonality in signals
- sufficient annual accumulation
- sufficient human resources for counting
- error accumulates

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  - model of accumulation:
    - estimated from isotopic content of ice
  - model of mechanical processes after accumulation:
    - i.e. snow densification, ice flow
    - poorly known parameters

- comparison with other dated records
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  - uncertainty in other record
  - uncertainty in link between records

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⇒ quantify uncertainty in the accumulation model and derive
  the dating uncertainty incorporating other dating methods
Existing Dating Methods

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the dating uncertainty incorporating other dating methods
A Bayesian Approach to Glaciological Modelling

- Glaciological Model and its Uncertainty
- The Bayesian Approach
- Typical Set of Prior Assumptions
- Sampling from the Posterior
- Calculating the Marginal Posterior

The Dating Uncertainty

Discussion

A Bayesian Approach to Glaciological Modelling
Glaciological Model and its Uncertainty

Accumulation Model

chemical measurement: isotopic content of ice → annual accumulation rate → times covered by each slice of the core
Accumulation Model

chemical measurement: isotopic content of ice $\rightarrow$ annual log-accumulation rate $\rightarrow$ times covered by each slice of the core

$Y \rightarrow f(Y) \rightarrow T = \frac{g(D)}{ef(Y)}$

deformation of depths
**Glaciological Model and its Uncertainty**

**Accumulation Model**

chemical measurement: isotopic content of ice → annual log-accumulation rate → times covered by each slice of the core

\[ Y \xrightarrow{} f(Y) \xrightarrow{} T = \frac{g(D)}{e^{f(Y)}} \]

**Its Uncertainty**

\[ Y \xrightarrow{} f_\theta(Y) = A + \varepsilon \xrightarrow{} T = \frac{g(D)}{e^A} \]

‘true’ log-accumul. rates

‘observed’ log-accumul. rate

\( f \) involves uncertain parameters \( \theta \)

(model error)

(\text{inverse modelling: Parrenin et al., 2001})
The Bayesian Approach

Quantities of interest
- latent variable $A$
- parameters $\theta, \sigma$ (where $\varepsilon_i \sim N(0, \sigma^2)$ iid)

\[
f_\theta(Y) = A + \varepsilon
\]

\[
P(A, \theta, \sigma | \cdot)
\]
The Bayesian Approach

- Quantities of interest
  - latent variable $A$
  - parameters $\theta, \sigma$ (where $\varepsilon_i \sim N(0, \sigma^2)$ iid)
- Sources of information
  - ice core measurements $Y$ (data)

$f_\theta(Y) = A + \varepsilon$

$$P(A, \theta, \sigma|Y),$$
The Bayesian Approach

- Quantities of interest
  latent variable $A$
  parameters $\theta$, $\sigma$ (where $\varepsilon_i \sim N(0, \sigma^2)$ iid)

- Sources of information
  ice core measurements $Y$ (data)
  prior knowledge on quantities of interest
  - recent weather records: not applicable
  - layer counted ice cores
    - assume accumulation model $f$ holds ‘globally’:
      fit a hierarchical linear model to 11 layer counted ice cores
      $\rightarrow$ use results as priors for parameters $\theta$ and $\sigma$

\[
P(A, \theta, \sigma|Y) \propto P(f_\theta(Y)|A, \theta, \sigma)P(\sigma)P(\theta)P(A)
\]
The Bayesian Approach

\[ f_\theta(Y) = A + \varepsilon \]

\[ r^2 \text{ iid} \]

`est licable`

\[ \text{holds `globally':} \]
\[ \circ \text{11 layer counted ice cores} \]
\[ \circ \text{ameters} \theta \text{ and} \sigma \]

\[ P(A, \theta, \sigma | Y, \propto P(f_\theta(Y) | A, \theta, \sigma)P(\sigma)P(\theta)P(A) \]
The Bayesian Approach

\[ f_\theta(Y) = A + \varepsilon \]

- **Quantities of interest**
  - latent variable \( A \)
  - parameters \( \theta, \sigma \) (where \( \varepsilon_i \sim N(0, \sigma^2) \) iid)

- **Sources of information**
  - ice core measurements \( Y \) (data)
  - prior knowledge on quantities of interest
    - recent weather records: not applicable
    - layer counted ice cores
      - assume accumulation model \( f \) holds ‘globally’:
        - fit a hierarchical linear model to 11 layer counted ice cores
        - use results as priors for parameters \( \theta \) and \( \sigma \)
      - use layer counting of top part as additional constraint

\[
P(A, \theta, \sigma | Y, L) \propto P(f_\theta(Y) | A, \theta, \sigma) P(\sigma) P(\theta) P(A) P(L | A)
\]
Typical Set of Prior Assumptions

Accumulation prior

\[ A_i \sim N(-0.5, 1.5^2) \]
Typical Set of Prior Assumptions

**Accumulation prior**

- $A_i \sim \mathcal{N}(-0.5, 1.5^2)$

**Accumulation model priors ($f$ linear)**

- $A_0 \sim \mathcal{N}(-0.79, 0.27^2)$
- Evidence from 11 cores

- $b \sim \mathcal{N}(0.027, 0.0037^2)$
- Evidence from 11 cores
Typical Set of Prior Assumptions

Accumulation prior

Accumulation model priors \( (f \text{ linear}) \) [Johnsen et al., 1995]

- Accumulation prior
  \[ A_i \sim N(-0.5, 1.5^2) \]

- Accumulation model priors
  \[ A_0 \sim N(-0.79, 0.27^2) \]

  Evidence from 11 cores

- Model error prior
  \[ \sigma \sim \Gamma(7355, 3.5\times10^{-05}) \]

  Evidence from 11 cores

Annual log–accum. rate \( A_i \) in m

Intercept \( A_0 \)

Slope \( b \)
Typical Set of Prior Assumptions

**Accumulation prior**

\[ A_i \sim N(-0.5, 1.5^2) \]

**Accumulation model priors (linear)**

\[ A_0 \sim N(-0.79, 0.27^2) \]

*Evidence from 11 cores*

\[ b \sim N(0.027, 0.0037^2) \]

*Evidence from 11 cores*

**Model error prior**

\[ \sigma \sim \Gamma(7355, 3.5e^{-0.05}) \]

*Evidence from 11 cores*

**Layer counting likelihood**

\[ P(L_i - A_i | A_i) = \frac{9}{10} \cdot N(0, 0.11^2) + \frac{1}{20} \cdot N(\log(1.8), 0.11^2) + \frac{1}{20} \cdot N(\log(0.6), 0.11^2) \]
Sampling from the Posterior

- direct calculation of \( P(A, \theta, \sigma | Y, L) \) intractable
Sampling from the Posterior

- direct calculation of $P(A, \theta, \sigma|Y, L)$ intractable
- traditional MCMC sampling: single-site updating (highly dependent parameters lead to bad performance)
Sampling from the Posterior

- direct calculation of $P(A; \theta, \sigma|Y, L)$ intractable
- traditional MCMC sampling: single-site updating
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- block-updating MCMC ([Rue and Held, 2005]):
  1) $\theta$
  2) $\sigma$
Sampling from the Posterior

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- block-updating MCMC ([Rue and Held, 2005]):
  1) $\theta$
  2) $A \sim P(A|\theta, \sigma, Y, L)$
Sampling from the Posterior

- direct calculation of \( P(A, \theta, \sigma | Y, L) \) intractable
- traditional MCMC sampling: single-site updating (highly dependent parameters lead to bad performance)
- block-updating MCMC ([Rue and Held, 2005]):
  1) \( \theta^* \) from some symmetric proposal
     \( \sigma^* \) from some symmetric proposal
  2) \( A^* \sim P(A | \theta^*, \sigma^*, Y, L) \)
  3) accept jointly with probability \( \alpha = \min \left\{ 1, \frac{P(\theta^*, \sigma^* | Y, L)}{P(\theta, \sigma | Y, L)} \right\} \)
Sampling from the Posterior

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  3) accept jointly with probability \( \alpha = \min \left\{ 1, \frac{P(\theta^*, \sigma^*|Y, L)}{P(\theta, \sigma|Y, L)} \right\} \)

where \( P(\theta, \sigma|Y, L) = \frac{P(A, \theta, \sigma|Y, L)}{P(A|\theta, \sigma, Y, L)} \)
Sampling from the Posterior

- direct calculation of $P(A, \theta, \sigma | Y, L)$ intractable
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     where $P(\theta, \sigma | Y, L) = \frac{P(A, \theta, \sigma | Y, L)}{P(A | \theta, \sigma, Y, L)}$
  4) repeat steps 1) - 3)
Sampling from the Posterior

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$\Rightarrow$ very efficient
Sampling from the Posterior

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  4) repeat steps 1) - 3)

$\Rightarrow$ very efficient
Calculating the Marginal Posterior

\[ P(\mathbf{A}|\theta, \sigma, \mathbf{Y}, \mathbf{L}) \]

\[ P(\mathbf{A}|\theta, \sigma, \mathbf{Y}, \mathbf{L}) \propto P(\mathbf{Y}|\mathbf{A}, \theta, \sigma)P(\mathbf{A}) \]
Calculating the Marginal Posterior

\[ P(A|\theta, \sigma, Y, L) \]

\[ P(A|\theta, \sigma, Y, L) \propto P(Y|A, \theta, \sigma)P(A) \]

\[ \propto N_{f_\theta(Y)}(A, \sigma^2) N_A(\mu_A, \sigma^2_A) \]

\[ \propto N_A\left( \frac{\mu_A\sigma^2 + f_\theta(Y)\sigma^2_A}{\sigma^2 + \sigma^2_A}, \frac{\sigma^2\sigma^2_A}{\sigma^2 + \sigma^2_A} \right) \]
Calculating the Marginal Posterior

\[
P(A|\theta, \sigma, Y, L) 
\]

\[
P(A|\theta, \sigma, Y, L) \propto P(Y|A, \theta, \sigma)P(A)P(L|A)
\]

\[
\propto f_{\theta,Y}(A, \sigma^2) N_A(\mu_A, \sigma_A^2)
\]

\[
\propto N_A \left( \frac{\mu_A \sigma^2 + f_{\theta,Y}(Y) \sigma_A^2}{\sigma^2 + \sigma_A^2}, \frac{\sigma^2 \sigma_A^2}{\sigma^2 + \sigma_A^2} \right)
\]
Calculating the Marginal Posterior

\[ P(A|\theta, \sigma, Y, L) \propto P(Y|A, \theta, \sigma)P(A)P(L|A) \]

\[ \propto N_{f_\theta(Y)}(A, \sigma^2) N_A(\mu_A, \sigma_A^2) N_L(A, \sigma_L^2) \]

\[ \propto N_A \left( \frac{\mu_A \sigma_A^2 + f_\theta(Y) \sigma_A^2}{\sigma^2 + \sigma_A^2}, \frac{\sigma^2 \sigma_A^2}{\sigma^2 + \sigma_A^2} \right) N_L(A, \sigma_L^2) \]
Calculating the Marginal Posterior

\[ P(A | \theta, \sigma, Y, L) \propto P(Y | A, \theta, \sigma) P(A) P(L | A) \]

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\[ \left\{ \begin{array}{l}
N_A \left( \frac{\mu_A \sigma^2 + f_{\theta}(Y) \sigma_A^2}{\sigma^2 + \sigma_A^2}, \frac{\sigma^2 \sigma_A^2}{\sigma^2 + \sigma_A^2} \right) \\
N_A \left( \frac{(\mu_A \sigma^2 + f_{\theta}(Y) \sigma_A^2) \sigma_L^2 + (L \sigma_A^2 + \sigma_A^2 \sigma_L^2)}{(\sigma^2 + \sigma_A^2) \sigma_L^2 + \sigma^2 \sigma_A^2}, \frac{\sigma^2 \sigma_A^2 \sigma_L^2}{(\sigma^2 + \sigma_A^2) \sigma_L^2 + \sigma^2 \sigma_A^2} \right)
\end{array} \right. \]

if no LC exists.

else
Calculating the Marginal Posterior

\[ P(A|\theta, \sigma, Y, L) \propto P(Y|A, \theta, \sigma)P(A)P(L|A) \]

\[ \propto N_{f_\theta(Y)}(A, \sigma^2)N_A(\mu_A, \sigma_A^2) \sum_{j=1}^{3} w_j N_L(A+a_i, \sigma_L^2) \]

\[ \propto N_A \left( \frac{\mu_A \sigma^2 + f_\theta(Y) \sigma_A^2}{\sigma^2 + \sigma_A^2}, \frac{\sigma^2 \sigma_A^2}{\sigma^2 + \sigma_A^2} \right) \sum_{j=1}^{3} w_j N_L(A+a_i, \sigma_L^2) \]

\[ \propto \left\{ \begin{array}{ll}
N_A \left( \frac{\mu_A \sigma^2 + f_\theta(Y) \sigma_A^2}{\sigma^2 + \sigma_A^2}, \frac{\sigma^2 \sigma_A^2}{\sigma^2 + \sigma_A^2} \right) & \text{if no LC exists,} \\
\sum_{j=1}^{3} N_A \left( \frac{(\mu_A \sigma^2 + f_\theta(Y) \sigma_A^2) \sigma_L^2 + (L-a_i) \sigma_A^2 \sigma^2}{(\sigma^2 + \sigma_A^2) \sigma_L^2 + \sigma_A^2 \sigma^2}, \frac{\sigma^2 \sigma_A^2 \sigma_L^2}{(\sigma^2 + \sigma_A^2) \sigma_L^2 + \sigma_A^2 \sigma^2} \right) & \text{else}
\end{array} \right. \]
The Dating Uncertainty
Dating Uncertainty in an Example

Toy example

A shallow core from Dyer Plateau, Antarctica ($70^\circ 39'S$, $65^\circ 01'W$)

Dating uncertainty

\[
\begin{align*}
\text{Estimated number of years} & \quad 0 & 20 & 40 & 60 & 80 \\
\text{Depth in m} & \quad 0 & 10 & 20 & 30 & 40 & 50
\end{align*}
\]

- $98.71 \pm 6.02$ years
- Block updating MCMC

(Rue and Held, 2005)
Toy example

A shallow core from Dyer Plateau, Antarctica (70°39’S, 65°01’W)

Dating uncertainty

![Graph showing estimated number of years versus depth in m for two different locations.](http://en.wikipedia.org/wiki/Image:Flag_of_Antarctica.svg)
Example Continued

Graphs showing estimated number of years against depth in meters for different time periods: 0, 10, 30, and 50 years. The graphs illustrate the dating uncertainty with posterior distributions (constrained by LC).
Posterior Distributions (constrained by LC)

Accumulation prior

Accumulation model priors ($f$ linear, Johnsen et al., 1995)

- Accumulation prior
  - $A_i \sim N(-0.5, 1.5^2)$

- Accumulation model priors ($f$ linear)
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Discussion
Current and Future Research

- current research
  - gain better prior knowledge
  - include mechanical model
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- current research
  - gain better prior knowledge
  - include mechanical model

- future research
  - more complex accumulation models $f$
  - multicore, multiproxy analysis
  - statistical approach for layer counting (pilot: J. Wheatley)
    → combine
Current and Future Research

- current research
  - gain better prior knowledge
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- future research
  - more complex accumulation models
  - multicore, multiproxy analysis
  - statistical approach for layer counting (pilot: J. Wheatley)
    → combine

- problems
  - hard to quantify uncertainty further back in time
  - hiatus: summer melting, ice flow disturbances
Thank you!
Dating Uncertainty in an Example Continued

147.08 ± 9.81 years

